USING CONTROLLED CHAOS FOR DIGITAL SIGNALING:
A PHYSICAL PRINCIPLE FOR 1-STAGE WAVEFORM SYNTHESIS

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We consider the possibility of using controlled chaotic operation to produce complex signals directly at the high-power level in a microwave source. By implementing *symbolic dynamics control*, a high-power source could be guided through complex pulsed, frequency hopped, or digital information bearing sequences with negligible control power. This possibility arises because chaotic behavior is naturally complex and sensitive to small perturbations, and we can gently guide the signal through a preexisting chaotic orbit with tiny controlling perturbations. We have done this in an audio-frequency circuit, and the basic control technique could be extended to high frequencies.

1. INTRODUCTION

We have devised a principle for signal generation based on the fundamental connection between chaos and information theory.[1] It is possible to use a (possibly high-powered) nonlinear oscillator to produce a complex signal in one step: A simple high power *chaotic* oscillator naturally produces complex signals, and small perturbing current pulses can gently guide the dynamics of the system through the desired output behavior. The high-power oscillator could thus remain simple and efficient, with the more complex control circuitry at the microelectronic level. We argue that this method is natural for producing high-power signals in general, because no amplification stages are required, and one can operate the source in a strongly nonlinear state.

This principle for signal generation is based on the mathematical formalism of *symbolic dynamics*.[2] Symbolic dynamics lets us formulate a description of chaos that is more akin to digital signal processing theory than it is to continuous-time signal theory. By assigning a discrete alphabet to the system state space using symbolic dynamics, the chaotic system becomes a symbol source, and because it is a continuous-time waveform source, it is also a digital signal source.

A very simple chaotic electrical oscillator, for example, can produce a seemingly random sequence of positive and negative (bipolar) voltage peaks.[3] If these bipolar peaks are assigned binary symbols 0 and 1 respectively, then the oscillator can be viewed as a binary digital source. We can control the binary sequence by using small perturbing current pulses, and thus control the large-scale waveform dynamics. (More sophisticated waveforms are possible with more complex systems.) In the same way that small-perturbation control can be used to guide a chaotic system through a pre-chosen periodic path[4] in state space, we can guide the system through a desired binary sequence.

The important point about using symbolic dynamics for general chaos control is that one need specify only one new symbol for each Poincaré crossing, and thus symbolic dynamics provides the crossover from discrete-time control (Poincaré sampling) to fully digital control (discrete alphabet). With the advent of high-speed digital signal processing chips, it should be possible to extend the basic control experiment that we describe here to microwave frequencies because of the minimal computation required for our control procedure. It is also possible to reduce the control described here to a fully binary approach that would require no multi-level A/D conversion. We point out during the description of our experiment some modifications that would simplify a high-frequency implementation.

2. THE LORENZ SOURCE

The Lorenz[6] system is the prototypical continuous-time system for symbolic dynamics control and communication. The simplest information source is *the binary symmetric* (*memoryless*) *source*, a source of binary symbols (bits) in which each bit is equally likely ($p(0)=p(1)=1/2$) and the probabilities are independent of the past. The Lorenz system approximates the binary symmetric source, and it is also a waveform source. Although the correspondence is not perfect, it is remarkable considering the origins of the Lorenz system.[6]

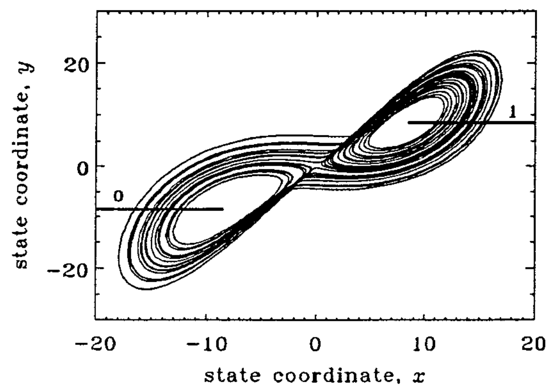

Fig. 1. Lorenz trajectory showing Poincaré surface and binary partition.

The Lorenz equations are $\dot{x} = -\sigma x + \sigma y$, $\dot{y} = Rx - y - xz$, and $\dot{z} = -bz + xy$, with parameter values $\sigma = 10$, $R = 28$, and $b = 8/3$. The state point (x, y, z) moves on a chaotic attractor in the three-dimensional state space. A trajectory of the Lorenz system is shown in Fig. 1. We have labeled one lobe of the attractor with the symbol 0, and the other with a 1. The sequence of lobe cycles thus defines a binary sequence. For the standard parameters, this binary sequence (somewhat roughly) approximates the statistics of a coin toss.

The lines shown intersecting the lobes in Fig. 1 are two half-planes viewed edge-on, and represent the two branches of a Poincaré surface of section. The half-planes are defined by the equations $y = \pm\sqrt{b(R-1)}$ and $|x| \geq \sqrt{b(R-1)}$. Because the Lorenz equations represent a highly dissipative flow, the intersection of the attractor with the Poincaré surface is approximately a single thin arc, and we can define a surface of section coordinate as $\xi = |x| - \sqrt{b(R-1)}$.

Because the system is deterministic, the symbol sequence produced after the state point

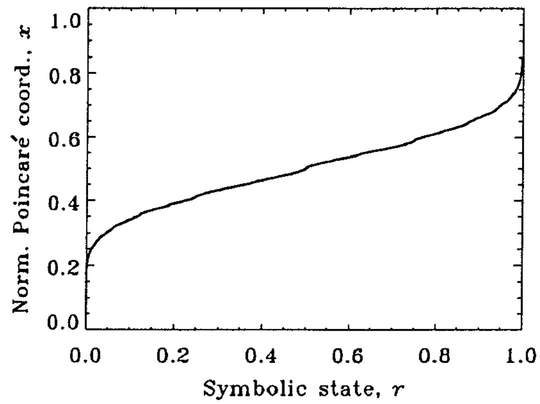

Fig. 2. Lorenz binary inverse coding function $x(r)$.

crosses the surface of section is determined by the point of crossing. Suppose the state point passes through the 0 branch of the surface of section and generates the binary symbol sequence $b_1b_2b_3$... . We use the *binary fraction* to represent this symbol sequence by the real number $0.b_1b_2b_3...$, where the n^{th} place behind the decimal point has value 2^{-n}. We refer to this real number $r = \sum_1^\infty b_n 2^{-n}$, specifying the future symbol sequence, as the *symbolic state* of the system. This defines a function $r(x)$ relating the symbolic state to the Poincaré coordinate, which we call the statepoint. The inverse of this function $x(r)$ for the Lorenz system is shown in Fig. 2. In this figure the ξ coordinate on the surface of section has been used as a the statepoint, and has been normalized so that $x = \xi / L$. (The statepoint x is of course not the same as the state coordinate x in the Lorenz equations.) We call this function the *symbolic* *waveform coding function*, or just the *coding function*. (Mapping discrete symbols to basis functions is sometimes called waveform coding.)

In practice, we store values of r to finite precision in a *symbol register*, say ten bits long. To produce a desired sequence, we first *set* $r_0 = r(x_0)$, so that the first finite-length desired symbolic state is determined by x_0, the value of x on the first pass through the Poincaré surface. After the first pass through the surface, the bit pattern stored in the symbol register is shifted to the left, and the most-significant bit, representing the symbol just produced, is discarded. The first bit of the desired symbol string is then placed into the least-significant-bit slot of the symbol register. The new value of r is then used as the desired symbolic state, and the state coordinates are corrected (one can simply perturb the state coordinates to the correct values) to correct the symbolic state when the state point next crosses the surface of section. (The bitwise complement of the bit pattern in the symbol register is used for lobe 1.) This process is repeated indefinitely to control the desired binary sequence.

Now, in Fig. 1, the oscillations about the 0 and 1 attractor lobes can be seen to correspond to negative and positive maxima, or spikes, in the x projection of the full state point. If this x projection $x(t)$ is used as the transmit signal, then the message can be extracted (for example) by simply observing the sequence of spikes in the waveform. The Lorenz system offers probably the simplest example of how symbolic dynamics can be used to transmit digital information.

Figure 1 is actually a controlled Lorenz trajectory, an ASCII encoding of the word "chaos."

3. SYMBOLIC CONTROL EXPERIMENT

We now summarize an audio frequency experiment. Fig. 3 shows the trajectory of the double scroll oscillator circuit[7] tuned to produce a Rössler band. We chose this region of operation for a first experiment because the symbolic dynamics is simple.[8] The Poincaré surface is shown, as well as the locus at which control pulses occur on the attractor (the white swath). We apply tiny current pulses into a capacitor every cycle to correct the trajectory.[8]

We have encoded (in 7-bit ASCII) the message, "Yea, verily, I say unto you: A man must have Chaos yet within him to birth a dancing star. I say unto you: You have yet Chaos in you." – Friedrich Nietzsche, *Thus Spake Zarathustra.* The simple code $0 \to 01$, $1 \to 11$ imposes a runlength limit of 1 on 0's which satisfies the grammar for the system.[8] To demonstrate the effect of the information source statistics on the signal, we a random sequence of

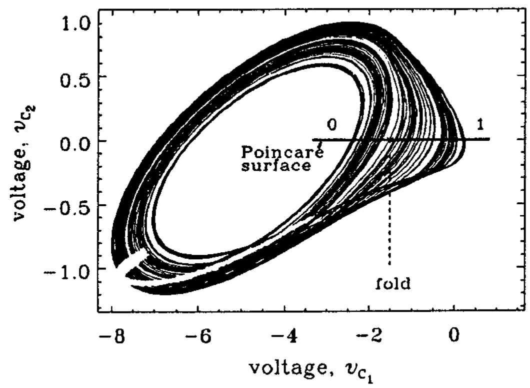

Fig. 3. Experimental Rössler band with Poincaré surface, symbolic partition, and locus of controls.

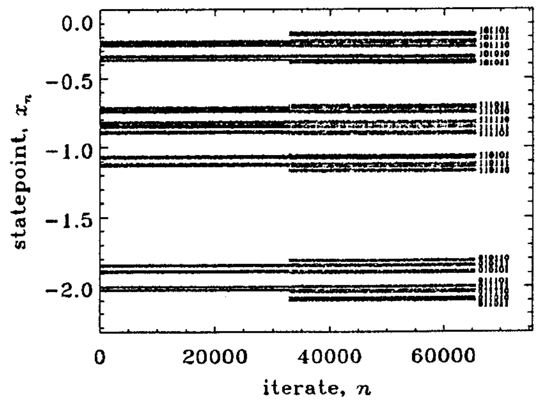

Fig. 4. Controlled transmission of two statistically different binary sequences.

0's and 1's with the same runlength restriction. A statepoint sequence corresponding to the repeated transmission of the Nietzsche quotation followed (starting at cycle $n \cong 33000$) by transmission of the random bit stream is shown in Fig. 4. (The rms control current during the whole sequence was 0.2 μA; circuit currents are a few milliamps.) Because of the constraint imposed by the code, the statepoints x_n fall within bands on the Poincaré surface: The signal exists on a finite-resolution Cantor set. The bands correspond to more than one binary symbol. We can resolve up to six bits, and have labeled the six-bit sequences on Fig. 4. This means that one can extract several bits from one sample, given a sufficient signal to noise ratio. The overly-constrained *symbol* sequence constrains the *statepoint* sequence to a point set of fractional capacity dimension. Because the binary sequence representing the quotation is *more* restrictive than the random bit sequence (the code $0 \rightarrow 01$, $1 \rightarrow 11$ is more restrictive), the signal is confined to *narrower* bands during the quotation than during the random bit sequence.

Figure 5 shows a statepoint sequence generated by controlling the oscillator symbolic dynamics through a sequence of periodic orbits without dropping control in between. We can also reduce the dwell time to less than ten cycles, which causes switches so rapidly that the periodic orbits are essentially pulses heard as a sequence of pops on a loudspeaker.

Several simplifications could be used to implement this control procedure at microwave frequencies. First, the control pulses need not have a great deal of resolution, in fact, one could use on/off control to correct the trajectory only after the error grows past a predetermined value, or the pulse-on time could be modulated. Furthermore, a reference voltage and a low-resolution sample relative to the reference could be used to determine the error. We are developing a technique that uses a 1-bit sample to determine a go/no-go condition for firing a control pulse of fixed amplitude. Thus fast multi-level A/D converters are not needed. We therefore do not think that it would be very difficult to repeat this experiment at microwave frequencies.

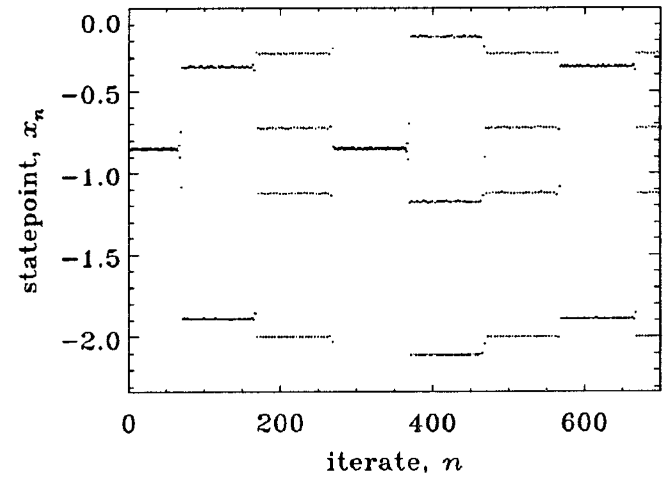

Fig. 5. Controlled "orbit hopping." Each periodic orbit is visited for about 100 cycles.

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